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Correspondence-Principle Condition on the Reissner–Nordström Solution for Charged Leptons

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Current experimental evidence supports the picture of three charged leptons ($e^{\pm}, \mu^{\pm}, \tau^{\pm}$), a triplet with mass splitting that bears a resemblance to the Gell–Mann–Okubo form. To elucidate the overall mass scale, a charged lepton is viewed as a mass point that engenders a local Reissner–Nordström space-time geometry, and the Einstein–Maxwell action is evaluated through an invariant space-time region associated with the particle's radiation reaction interval $2e^2/3m$. What emerge for the electron, muon, and tau are values of the Einstein–Maxwell action in the neighborhood of $\hbar$. The mean value of the three action integrals is $\bar{A}=(140.93)e^2=(1.0284)\hbar$, and thus the apparent quantum condition $\bar{A}\sim\hbar$ sets the mass scale for the three charged leptons.

The mass ratios for the electron, muon, and tau are given by the semitheoretical formula (Nambu, 1952; Rosen, 1964; Barut, 1979)

$$m=m_e+\hbar\hat{r}_e^{-1}F=m_e\left(1+\tfrac{3}{2}\alpha^{-1}F\right) \tag{1}$$

where m_e is the mass of the electron, $\hat{r}_e=2e^2/3m_e$ is the classical radiation reaction interval (Dirac, 1938), and the quantum number F has the values

$$F_e=0, \qquad F_\mu=1, \qquad F_\tau=17 \tag{2}$$

Now known rather accurately up to 30 GeV, the e^+–e^- total cross section appears to preclude existence of any additional heavier charged leptons (Perl, 1979). Assuming that the charged leptons are just three in number and constitute a triplet, one can introduce an operator T_3 with eigenvalues

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+ 1,0, − 1 and eigenstates identified with the physical particles:

$$T_3|e^\pm\rangle = 0, \quad T_3|\mu^\pm\rangle = -|\mu^\pm\rangle, \quad T_3|\tau^\pm\rangle = |\tau^\pm\rangle \tag{3}$$

Then the mass operator which yields (1) with (2) follows by setting

$$F = \left[1 + (T_3 + 1)^4\right] T_3^2, \tag{4}$$

a form somewhat suggestive of Gell–Mann–Okubo splitting, where T_3 may be interpreted as the third component of "leptospin." What is particularly noteworthy regarding formula (1) with (4) is that the lepton mass splitting is given exclusively in terms of the quantum electrodynamics constant α, notwithstanding the fact that the leptons are coupled to hadrons (or quarks) via the electromagnetic and weak interactions. However, because m_e must be prescribed empirically, the overall mass scale for the charged leptons is not fixed by formula (1)[1].

Since experiments show the electron to be essentially pointlike and without measurable spatial extension on a millifermi scale, the classical Einstein–Maxwell field theory for an electrically charged mass point may in fact apply in a correspondence sense to the electron, the muon, and the tau. That is, in the neighborhood of a charged lepton at rest the local Riemannian-space-time geometry is given by the Reissner–Nordström line element (units such that $c = 1$)[2]

$$ds^2 = -\Omega (dt)^2 + \Omega^{-1}(dr)^2 + r^2\left[(d\theta)^2 + \sin^2\theta (d\phi)^2\right] \tag{5}$$

where $\Omega \equiv 1 - 2Gmr^{-1} + Ge^2r^{-2}$ with $G = 6.673(\pm 0.003) \times 10^{-8}$ cm^3 g^{-1} sec^{-2} denoting Newton's constant and e denoting the observable unit of electric charge. Because $G^{1/2}m/e$ is less than or of the order 10^{-18} for the

[1]It is quite possible that the mean mass of the charged leptons can be computed in a perturbation-theoretic manner from QED with bare-mass zero and a quantum-gravitational cutoff (e.g., Fryberger, 1979; Rosen, 1971). Nevertheless, it is not uncommon for fundamental phenomena in the quantum domain to admit manifestly different but independently correct descriptions, which later appear to be compatible or equivalent in the context of deeper theory. This, and the need for an alternative (nonperturbative) approach to the self-mass problem, justifies the consideration of the Reissner–Nordström solution in the present communication. That QED itself actually admits application of the correspondence principle to the lepton self-mass problem has in fact been shown by Fomin (1976).

[2]The space-time coordinates used here are of direct geometrical significance (Rosen, 1962), and therefore have an immediate observational meaning. In particular, t is the time measured by a clock at rest far from the charged mass-point, and r is given at any point in terms of the local Ricci curvature quadratic invariant: $\frac{1}{2}(R^{\mu\nu}R_{\mu\nu})^{1/2} = Ge^2r^{-4}$.

masses of interest, one has[3]

$$\Omega = 1 + Ge^2r^{-2}$$

with the Schwarzschild term trivial in relative magnitude *for all r*. The electromagnetic field has the nonvanishing radial component $f_{01} = -f_{10} = \pm er^{-2}$ (Weyl, 1922; index notation: $x_0 = t$, $x_1 = r$), and thus from (5)

$$g^{\mu\nu}g^{\rho\sigma}f_{\mu\rho}f_{\nu\sigma} = -2e^2r^{-4} \tag{7}$$

Consider the stationary value of the Einstein–Maxwell action

$$A \equiv \frac{1}{16\pi}\int\left(G^{-1}R - g^{\mu\nu}g^{\rho\sigma}f_{\mu\rho}f_{\nu\sigma}\right)(-g)^{1/2}d^4x \tag{8}$$

for (5) with $\triangle t = \int dx_0 = 2\pi r\Omega^{-1/2}$, the coordinate time interval for light to travel around a circle of constant radius r [according to (5) with $ds^2 = 0$], and $r \leqslant \hat{r} \equiv 2e^2/3m$, corresponding to a spherical spatial region of radius[4] equal to the particle's radiation reaction interval. Because the scalar curvature $R = 0$ for all $r > 0$ and $\triangle t$ is asymptotic to $2\pi G^{-1/2}e^{-1}r^2$ as $r \to 0$ (giving $R\triangle t = 0$ for all $r \geqslant 0$, in spite of the δ function singularity in R at $r = 0$) it follows by putting (5)–(7) into (8) that

$$A = \frac{1}{8\pi}\int_0^{\hat{r}}(e^2r^{-4})(2\pi r\Omega^{-1/2})(4\pi r^2dr) \tag{9}$$

$$= \pi e^2\int_0^{\hat{r}}(r^2 + Ge^2)^{-1/2}dr = \pi e^2\left\{\ln\left[(1 + G^{-1}e^{-2}\hat{r}^2)^{1/2}\right.\right.$$

$$\left.\left. + G^{-1/2}e^{-1}\hat{r}\right]\right\} = \pi e^2\left[\ln(2G^{-1/2}e^{-1}\hat{r}) + O(Ge^2\hat{r}^{-2})\right]$$

Since $Ge^2\hat{r}^{-2}$ is less than or of the order 10^{-36} for the leptons, the final member of (9) yields

$$A = \pi e^2\left[\ln(4e/3G^{1/2}m)\right]$$

$$= \begin{cases} 155.055e^2 \text{ for the electron with } m = 0.5110034 \text{ MeV} \\ 138.305e^2 \text{ for the muon with } m = 105.6595 \text{ MeV} \\ 129.429e^2 \text{ for the tau with}^5\ m = 1782 \text{ MeV} \end{cases} \tag{10}$$

[3]Of course the Schwarzchild term $2Gmr^{-1}$ cannot be dropped for all r in the Christoffel symbols and associated curvature tensors, and (6) can only be used in *algebraic expresions* that involve the metric tensor.

[4]From the line element (5) it follows that the observable *proper radius* is $\triangle s = \int_0^{\hat{r}}\Omega^{-1/2}dr = \hat{r}$, with omission of a numerically trivial term (relative order 10^{-18} or less for the leptons).

[5]Tau mass value according to Perl (1979).

The accuracy of the A values in (10) for the electron and muon is limited by the experimental uncertainty in G, which produces a systematic $\delta A \cong \pm 7 \times 10^{-4} e^2$. All of the values in (10) are in the neighborhood of $\hbar$, and the mean value of the three action integrals is precisely

$$\bar{A} = (140.93)e^2 = (1.0284)\hbar \tag{11}$$

As shown empirically by (11), the correspondence-principle condition $\bar{A} \sim \hbar$ serves to fix the mass scale for the leptons. Notice that the condition $\bar{A} \sim \hbar$ would still obtain if $\hat{r}$ in (9) were set equal to either e^2/m or $e^2/2m$ (rather than $2e^2/3m$) because of the insensitivity of the logarithm.

The apparent quantum condition $\bar{A} \sim \hbar$ should be derivable in a future theory, along with the leptonic mass splitting shown in (1). In the Feynman path integral quantization of Einstein–Maxwell theory, the action (8) enters through the amplitude factor (exp $iA/\hbar$), and it was pointed out many years ago (Wheeler, 1954) that field histories with $A \sim \hbar$ interfere constructively in the Feynman sum and can make a dominant contribution to the probability amplitude for time evolution of the state. To achieve correspondence-principle agreement, the action must be evaluated for the Reissner–Nordström solution through an invariant space-time region associated with the particle's radiation reaction interval $\hat{r} = 2e^2/3m$, as carried out in (9). Presumably as a residue of this, $\hat{r}_e$ appears in the mass splitting formula (1).

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